# Rings of Differential Operators

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These notes are used to record what I have learned about differential operator rings in the summer vacation of 2023. Main references are [MR87] and [Lam01]. There might be some typos and mistakes in these notes. Please let me know if you see one. Thanks!

## Contents



## Introduction

## 1 Basic Examples

### 1.1 Skew Polynomial Rings

In this section, we shall study the "twisted" verison of the polynomial ring, which is called the skew polynomial ring. First consider the set of polynomials over a unital ring $R$, say $X = \{a_0 + a_1x + \cdots + a_nx^n | a_i \in R, 0 \le i \le n, n \in \mathbb{N}\}$. Clearly, $X$ has a natural additive group structure. Next, we shall define the multiplication over $X$. In order that degrees behave appropriately, that is, one needs the fact that $\deg f(x)g(x) \le \deg f(x) + \deg g(x)$ for all $f(x), g(x) \in X$, it is required that $xa \in Rx + R, \forall a \in R$. So we may write $xa = \sigma(a)x + \delta(a)$. It is natrual to hope that $x(ab) = (xa)b, \forall a, b \in R$, which implies that $\sigma(ab) = \sigma(a)\sigma(b), \forall a, b \in R$ and $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. Motivated by the discussion above, we now give the following definition.

**Definition 1.1.** Let $R$ be a unital ring and $\sigma$ be a unital ring endomorphism of $R$. An endomorphism $\delta \in \mathrm{End}(R, +)$ is said to be a **$\sigma$-derivation** if $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$.

**Remark.** Clearly, for a $\sigma$-derivation $\delta$, one has $\delta(1) = 0$.

Given a unital ring $R$, an endomorphism $\sigma$ and a $\sigma$-derivation $\delta$, it is possible to construct a polynomial ring as described above. Put $E = \mathrm{End}(R^{\mathbb{N}}, +)$, then one has a natural embedding $R \to E, a \mapsto a_l$, where $a_l$ denotes the acting by left multiplication. One also has an element $\psi \in E$ defined by $\psi(a_i)_{i=0}^{\infty} = (\sigma(a_{i-1}) + \delta(a_i))_{i=0}^{\infty}$, where $a_{-1} = 0$. It is a quick check that $\psi a_l = \sigma(a)_l\psi + \delta(a)_l$ for all $a \in R$.

Now put $\Theta : X \to E, a_0 + a_1x + \cdots + a_nx^n \mapsto (a_0)_l + (a_1)_l\psi + \cdots + (a_n)_l\psi^n$, we shall give $X$ a unital ring structure via $\Theta$. Clearly, $\Theta$ is an additive group morphism and $\Theta(1) = \mathrm{id}_{R^{\mathbb{N}}}$. A basic observation is that

**Claim.** The map $\Theta$ is an injection.

*Proof.* Take any $a_0 + a_1x + \cdots + a_nx^n \in \mathrm{Ker}\Theta$, then $(a_0)_l + (a_1)_l\psi + \cdots + (a_n)_l\psi^n = 0$. So

$$((a_0)_l + (a_1)_l\psi + \cdots + (a_n)_l\psi^n)(1, 0, 0, \cdots) = (a_0, a_1, a_2, \cdots),$$

which shows that $a_0 = a_1 = \cdots = a_n = \cdots = 0$. □

Since $\psi a_l = \sigma(a)_l\psi + \delta(a)_l, \forall a \in R$, $\mathrm{Im}\Theta$ is closed under mutiplication and hence $\mathrm{Im}\Theta$ is a subring of $E$. Thus one can define

$$f \cdot g = \Theta^{-1}(\Theta(f)\Theta(g)), \forall f, g \in X$$

to make $X$ into a unital ring, which is called the **skew polynomial ring** and is denoted by $R[x; \sigma, \delta]$.

By definition, $xa = \sigma(a)x + \delta(a), \forall a \in R$ in $R[x; \sigma, \delta]$. It is obvious that $R[x; \sigma, \delta]$ is a ring extension of $R$, thus $R[x; \sigma, \delta]$ is also called the **Ore extension** of $R$.

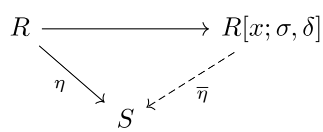
Suppose $\sigma$ is an automorphism, then each element in $R[x; \sigma, \delta]$ can be written in the form $\sum_{i=0}^{n} x^n a_n$ in a unique way. One can easily prove this by induction on $n$. The following notations will be used consistently throughout.

**Example 1.2.** Suppose $\sigma = \mathrm{id}_R$, $R[x; \sigma, \delta]$ is written as $R[x; \delta]$.

**Example 1.3.** Suppose $\delta = 0$, $R[x; \sigma, \delta]$ is written as $R[x; \sigma]$.

Clearly, $R[x] = R[x; \mathrm{id}_R, 0]$, hence the skew polynomial ring can be viewed as a generalization of the classical polynomial ring. And the Ore extension has the following universal property.

**Proposition 1.4.** The Ore extension $R[x; \sigma, \delta]$ of a unital ring $R$ has the universal property that if $\eta : R \to S$ is a unital ring homomorphism and $y \in S$ has the property that $y\eta(a) = \eta(\sigma(a))y + \eta(\delta(a))$ for all $a \in R$, then there is a unique ring homomorphism $\overline{\eta} : R[x; \sigma, \delta] \to S$ such that $\overline{\eta}(x) = y$ and the diagram




commutes.

*Proof.* Set $\overline{\eta} : R[x; \sigma, \delta] \to S, a_0 + a_1x + \cdots + a_nx^n \mapsto \eta(a_0) + \eta(a_1)y + \cdots + \eta(a_n)y^n$. A direct check shows that $\tilde{\eta}(ax)\tilde{\eta}(f) = \tilde{\eta}(axf), \forall f \in R[x; \sigma, \delta]$. Then one can show that $\tilde{\eta}(ax^n)\tilde{\eta}(f) = \tilde{\eta}(ax^nf), \forall f \in R[x; \sigma, \delta]$ by induction on $n$. Thus the $\tilde{\eta}$ is a ring homomorphism since it preserves addition. The uniqueness of $\tilde{\eta}$ is clear. □

Now we note some ring theoretic properties of the Ore extension.

**Theorem 1.5.** Let $R$ be a unital ring and $R[x; \sigma, \delta]$ be an Ore extension of $R$.
(1)If $\sigma$ is injective and $R$ is an integral domain, then $R[x; \sigma, \delta]$ is an integral domain.
(2)If $R$ is a division ring, then $R[x; \sigma, \delta]$ is a principle left ideal domain.
(3)If $\sigma$ is an automorphism and $R$ is a prime ring, then $R[x; \sigma, \delta]$ is a prime ring.
(4)If $\sigma$ is surjective and $R$ is right(left) Noetherian, then $R[x; \sigma, \delta]$ is right(respectively left) Noetherian.

*Proof.* (1)Take $f = a_0 + a_1x + \cdots + a_nx^n, g = b_0 + b_1x + \cdots + b_mx^m$ with $a_n, b_m$ nonzero, then $fg$ has degree $n + m$ and leading coefficient $a_n\sigma^n(b_m)$. Thus $fg \neq 0$.

(2)Clearly, we also has the division algorithm in $R[x; \sigma, \delta]$. Thus for each nonzero left ideal of $R[x; \sigma, \delta]$, take a monic polynomial in this left ideal with least degree, then one can easily check that this monic polynomial generates the left ideal. It follows that the left ideal is principle.

(3)Take $f = a_0 + a_1x + \cdots + a_nx^n, g = b_0 + b_1x + \cdots + b_mx^m$ with $a_n, b_m$ nonzero, then we must show that $fR[x;\sigma,\delta]g \neq 0$. Since $\sigma$ is injective, $\sigma^n(b_m) \neq 0$, and so $a_nR\sigma^n(b_m) \neq 0$. Take $c \neq 0 \in a_nR\sigma^n(b_m)$ and write $c = a_na\sigma^n(b_m)$. Since $\sigma$ is also surjective, there is $b \in R$ with $a = \sigma^n(b)$. It follows that $c = a_n\sigma^n(bb_m) \neq 0$. Thus $fbg \neq 0$, which implies $0 \neq fRg \subseteq fR[x;\sigma,\delta]g$ as desired.

(4)Note that this is a generalization of Hilbert basis theorem, we shall prove it in a similar way.

**Case 1.** Suppose $R$ is right Noetherian, we need to prove that $R[x;\sigma,\delta]$ is right Noetherian, too. Take any nonzero right ideal $B$ of $R[x;\sigma,\delta]$, it suffices to show that $B$ is finitely generated. Any element in $R[x;\sigma,\delta]$ has the form $\sum\limits_{i=0}^{n} a_ix^i$. For each natrual number $n$, let $I_n$ be the set of $b_n \in R$ such that there exists an element of the form

$$b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0 \in B.$$

Since $B$ is a right ideal and $\sigma$ is surjective, it is a quick check that $I_n$ is a right ideal in $R$. It is obvious that

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots.$$

Since $R$ is right Noetherian, there is a positive integer $N$ with $I_N = I_{N+1} = \cdots$. For each natural number $0 \leq k \leq N$, we may assume that $I_k$ has a finite generator set, say $\{b_k^1, b_k^2, ..., b_k^{s_k}\}$. Then we have polynomials $f_k^i = b_k^ix^k + g_k^i$, where $\deg g_k^i < k$ and $1 \leq i \leq s_k$. Now it is easy to check that the finite set of polynomials

$$\{f_0^1, ..., f_0^{s_0}, f_1^1, ..., f_1^{s_1}, ..., f_N^1, ..., f_N^{s_N}\}$$

generates $B$. It follows that $R[x;\sigma,\delta]$ is right Noetherian.

**Case 2.** The left case is similar. We leave it to the reader to check this. □

**Remark 1.6.** One shall notice that any surjective endomorphism of a Noetherian ring is injective.

For the Ore extension $R[x;\sigma,\delta]$, it is clear that $R[x;\sigma,\delta]/R[x;\sigma,\delta]x \cong R$ as a left $R[x;\sigma,\delta]$-module. Thus we obtain a presentation of $R[x;\sigma,\delta]$: $\rho : R[x;\sigma,\delta] \to \mathrm{End}(R,+)$. Since $xa = \sigma(a)x + \delta(a), \forall a \in R$, it follows that $\rho(x)a = \delta(a), \forall a \in R$. Clearly, $\rho(b) = b_l, \forall b \in R$.

**Exercise 1.1.** Let $R$ be a left Noetherian ring and $\sigma$ be a surjective ring endomorphism of $R$. Show that $\sigma$ is injective.

**Exercise 1.2.** By **Theorem 1.5**(2), the Ore extension of a division ring is a principle left domain. Show that the Ore extension of an integral domain may not be a principle left domain.

**Exercise 1.3.** Let $R$ be a unital ring. Show that $R$ is prime if and only if $R[x]$ is prime.

**Exercise 1.4.** Let $R$ be a unital ring and $P$ be a prime ideal of $R[x]$, show that $P \cap R$ is a prime ideal of $R$.

**Exercise 1.5.** Let $R$ be an integral domain with an irreducible submodule. Show that $R$ is a division ring.

**Exercise 1.6.** Let $R$ be a unital ring. Show that $R$ is a division ring if and only if any left $R$-module is free.

**Exercise 1.7.** Let $\Bbbk$ be a field and $A$ be a $\Bbbk$-algebra. Recall that the **Gelfand-Kirillov dimension** of $A$ is defined by

$$\mathrm{GKdim}A = \sup_V\{\limsup_{n\to\infty} \log_n \dim_{\Bbbk} V^n\},$$

where the supremum is taken over all finite-dimensional subspaces $V \subseteq A$ containing $1_A$. For example, by a direct computation, one can easily show that $\mathrm{GKdim}\Bbbk[x_1, ..., x_m] = m$. In general, one has $\mathrm{GKdim}A = \mathrm{k.dim}A$ for any commutative affine $\Bbbk$-algebra $A$, where $\mathrm{k.dim}A$ denotes the Krull dimension of $A$. Prove that

$$\mathrm{GKdim}A[x;\delta] \geq \mathrm{GKdim}A + 1, \forall \delta \in \mathrm{Der}_{\Bbbk}A,$$

where $\mathrm{Der}_{\Bbbk}A$ denotes the set of all $\Bbbk$-derivations of $A$. Furthermore, suppose any finite dimensional subspace of $A$ is contained in some $\delta$-stable affine subalgebra of $A$, then $\mathrm{GKdim}A[x;\delta] = \mathrm{GKdim}A + 1$.

## 1.2 Weyl Algebras

In this section, we introduce the notion of Weyl algebra, which is an example of an Ore extension. Throughout this section, $\Bbbk$ denotes a field and $\Bbbk\langle X\rangle$ denotes the free algebra on a set $X$. Now $A_n(\Bbbk)$ denotes the $\Bbbk$-algebra with $2n$ generators $x_1, ..., x_n, y_1, ..., y_n$ and relations

$$x_i y_j - y_j x_i = \delta_{ij}, \text{ the Kronecker delta,}$$

and

$$x_i x_j - x_j x_i = y_i y_j - y_j y_i = 0$$

for all $1 \leq i, j \leq n$. The algebra $A_n(\Bbbk)$ is called the $n$th **Weyl algebra** over $\Bbbk$, which first appeared in quantum mechanics as an algebra generated by position and momentum operators. When $n = 1$, the generators are written as $x, y$ rather than $x_1, y_1$. In this case, $A_1(\Bbbk) = \Bbbk\langle x, y\rangle/(xy - yx - 1)$. By definition, one has the following.

**Basic Observation.** The $n$th Weyl algebra is an affine algebra.

Now we consider an alternative description of the Weyl algebra. Consider the polynomial ring $R = \Bbbk[x_1, ..., x_n]$, and set the sequence of rings $R_0 = R, R_1 = R_0[y_1, -\partial/\partial x_1], R_{i+1} = R_i[y_{i+1}; -\partial/\partial x_{i+1}]$. Clearly, any element in $R_n$ can be expressed in the form

$$\sum_{k_1,...,k_n,j_1,...,j_n} c_{k_1\cdots k_n j_1\cdots j_n} x_1^{k_1}\cdots x_n^{k_n} y_1^{j_1}\cdots y_n^{j_n} \text{ (finite sum)}$$

uniquely. Then we show that the $\Bbbk$-algebra $R_n$ has generators which satisfy the relations which define the Weyl algebra.

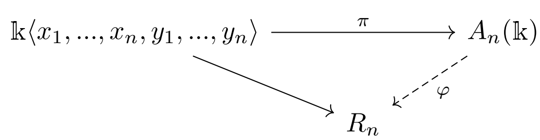
**Lemma 1.7.** Let $R_n$ be the $\Bbbk$-algebra defined as above. Then $R_n$ is generated by $\{x_1, ..., x_n, y_1, ..., y_n\}$ as a $\Bbbk$-algebra and one has $x_i y_j - y_j x_i = \delta_{ij}$, the Kronecker delta, and $x_i x_j - x_j x_i = y_i y_j - y_j y_i = 0$ for all $1 \leq i, j \leq n$.

*Proof.* By definition, $R_n$ is generated by $\{x_1, ..., x_n, y_1, ..., y_n\}$ and $x_i x_j = x_j x_i$ for all $1 \leq i, j \leq n$. Thus it remain to check that $x_i y_j - y_j x_i = \delta_{ij}$ and $x_i x_j - x_j x_i = y_i y_j - y_j y_i = 0$ for all $1 \leq i, j \leq n$. It is a drict computation that

$$x_i y_j - y_j x_i = \frac{\partial}{\partial x_j} x_i = \delta_{ij}.$$

Similarly, one can easily verify that $y_i y_j = y_j y_i$ for all $1 \leq i, j \leq n$. □

By the lemma above, one gets a natural algebra morphism $\varphi : A_n(\Bbbk) \to R_n$ such that the following diagram commutes.




It is clear that any element in $A_n(\Bbbk)$ can be written in the form

$$\sum_{k_1,...,k_n,j_1,...,j_n} c_{k_1\cdots k_n j_1\cdots j_n} x_1^{k_1}\cdots x_n^{k_n} y_1^{j_1}\cdots y_n^{j_n} + I,$$

where $I$ is the ideal of $\Bbbk\langle x_1, ..., x_n, y_1, ..., y_n\rangle$ generated by the relations which define the Weyl algebra. And

$$\varphi\Big(\sum_{k_1,...,k_n,j_1,...,j_n} c_{k_1\cdots k_n j_1\cdots j_n} x_1^{k_1}\cdots x_n^{k_n} y_1^{j_1}\cdots y_n^{j_n} + I\Big) = \sum_{k_1,...,k_n,j_1,...,j_n} c_{k_1\cdots k_n j_1\cdots j_n} x_1^{k_1}\cdots x_n^{k_n} y_1^{j_1}\cdots y_n^{j_n}.$$

It follows that $\varphi$ is an isomorphism. Thus one immediately obtains the following.

**Proposition 1.8.** Let $R_n$ be the $\Bbbk$-algebra defined as above. Then $R_n \cong A_n(\Bbbk)$ as $\Bbbk$-algebras.

**Remark.** Weyl algebra is an infinite dimensional algebra. One can also define Weyl algebra over a commutative unital ring $K$, then the above discussion still holds, one has $A_n(K) \cong R_n$ as $K$-algebras.

Hence we may identify $A_n(\Bbbk)$ with $R_n$. So any element in $A_n(\Bbbk)$ can be expressed in the form

$$\sum_{k_1,\dots,k_n,j_1,\dots,j_n} c_{k_1\cdots k_n j_1\cdots j_n} x_1^{k_1}\cdots x_n^{k_n} y_1^{j_1}\cdots y_n^{j_n} \text{ (finite sum)}$$

uniquely. Let us now record some basic ring theoretic properties of Weyl algebra.

**Proposition 1.9.** Let $A_n(\Bbbk)$ be the $n$th Weyl algebra. Then it is a Noetherian domain.

*Proof.* By **Theorem 1.5**(1), $A_n(\Bbbk)$ is an integral domain. And it is Noetherian by **Theorem 1.5**(4). □

**Remark.** Since any right Noetherian domain is a right Ore domain, $A_n(\Bbbk)$ has a right ring of quotients.

**Proposition 1.10.** Let $A_n(\Bbbk)$ be the $n$th Weyl algebra. Then it is neither left Artinian nor right Artinian.

*Proof.* We have already seen that any element in $A_n(\Bbbk)$ can be expressed in the form

$$\sum_{k_1,\dots,k_n,j_1,\dots,j_n} c_{k_1\cdots k_n j_1\cdots j_n} x_1^{k_1}\cdots x_n^{k_n} y_1^{j_1}\cdots y_n^{j_n} \text{ (finite sum)}$$

uniquely. Thus one has two descending chains

$$A_n(\Bbbk)y_n \supsetneq A_n(\Bbbk)y_n^2 \supsetneq A_n(\Bbbk)y_n^3 \supsetneq \cdots$$

and

$$x_1 A_n(\Bbbk) \supsetneq x_1^2 A_n(\Bbbk) \supsetneq x_1^3 A_n(\Bbbk) \supsetneq \cdots.$$

□

Next we'll show that the Weyl algebra over a field of characteristic 0 is simple. Before that, it is worthwhile to point out the following, which is a straightforward verification.

**Lemma 1.11.** For any $f \in A_n(\Bbbk)$, $x_i f - f x_i = \partial f/\partial y_i, y_i f - f y_i = -\partial f/\partial x_i$ for all $1 \le i \le n$.

A direct application of the lemma above leads to the following theorem.

**Theorem 1.12.** If $\operatorname{char}\Bbbk = 0$, then the $n$th Weyl algebra over $\Bbbk$ is a Noetherian simple domain.

*Proof.* We have already shown that $A_n(\Bbbk)$ is a Noetherian domain. So it remains to check that it is simple. Take any nonzero ideal of $A_n(\Bbbk)$, say $J$, and we pick a nonzero element

$$f = \sum_{k_1,\dots,k_n,j_1,\dots,j_n} c_{k_1\cdots k_n j_1\cdots j_n} x_1^{k_1}\cdots x_n^{k_n} y_1^{j_1}\cdots y_n^{j_n} \in J.$$

By the lemma above, both $\partial f/\partial x_i$ and $\partial f/\partial y_i$ belong to $J$ for $1 \le i \le n$. Consider the leading term of $f$ in the lexicographical sense, say $c_{k_1\cdots k_n j_1\cdots j_n} x_1^{k_1}\cdots x_n^{k_n} y_1^{j_1}\cdots y_n^{j_n}$, then $(k_1!k_2!\cdots k_n!j_1!j_2!\cdots j_n!)c_{k_1\cdots k_n j_1\cdots j_n} \in J$. Since $\operatorname{char}\Bbbk = 0$, we conclude that $1 \in J$ and hence $J = A_n(\Bbbk)$. □

**Remark.** The assumption on the characteristic of $\Bbbk$ is essential. In fact, if $\operatorname{char}\Bbbk = p > 0$, then in $A_1(\Bbbk)$, $x^m y - y x^m = m x^{m-1}, \forall m \ge 1$. Hence $x^p y = y x^p$. Thus $A_1(\Bbbk)x^p$ is a nonzero ideal $\subsetneq A_1(\Bbbk)$. Suppose $A_n(K)$ is the Weyl algebra over a commutative ring of characteristic zero, by the same reasoning, $A_n(K)$ is simple.

**Corollary 1.13.** If $\operatorname{char}\Bbbk = 0$, then any unital ring endomorphism of $A_n(\Bbbk)$ is injective.

In 1968, J. Dixmier asked the following, which is still an open problem today.

**Dixmier conjecture** ([Dix68])**.** If $\operatorname{char}\Bbbk = 0$, then any unital ring endomorphism of $A_n(\Bbbk)$ is an automorphism.

**Remark 1.14.** It can be shown that Dixmier conjecture holds for all $n \geq 1$ if and only if the Jacobian Conjecture holds for all $n \geq 1$[BKK05].

A classical result in commutative algebra is that any commutative Noetherian ring that has only finite number of prime ideals and all of these are maximal is Artinian. So it is natural to ask if the result holds in noncommutative setting. As an application of **Theorem 1.12**, we give a negative answer to this question.

**Corollary 1.15.** There is a Noetherian ring that is not Artinian but has only finite number of prime ideals and all of these are maximal.

*Proof.* Consider the Weyl algebra over a field of characteristic 0 is enough. □

We also show that Weyl algebra has no (nonzero) finite dimensional representations.

**Proposition 1.16.** Any nonzero module over Weyl algebra is infinite dimensional.

*Proof.* Suppose there is a nonzero finite dimensional module over $A_n(\Bbbk)$, say $V$. Then one has a natural $\Bbbk$-linear map $\rho : A_n(\Bbbk) \to \mathrm{End}_{\Bbbk}V, a \mapsto a_l$, which is clearly a $\Bbbk$-algebra morphism. Consider $x_1, y_1 \in A_n(\Bbbk)$, by definition, one has $x_1y_1 - y_1x_1 = 1$. Thus $\sigma(x_1)\sigma(y_1) - \sigma(y_1)\sigma(x_1) = \mathrm{id}_V$, which is absurd. Thus the result holds. □

**Example 1.17** (Weyl algebra and ring of differential operators)**.** Let $\Bbbk$ be a field of characteristic zero. Then $\Bbbk[x_1, ..., x_n]$ is a left $A_n(\Bbbk)$-module with $x_1, ..., x_n$ acting by multiplication and $y_i$ acting as $-\partial/\partial x_i$. A quick check shows that

$$-(x_i)_l \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j}(x_i)_l = \delta_{ij}$$

and

$$(x_i)_l(x_j)_l - (x_j)_l(x_i)_l = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} = 0$$

in $\mathrm{End}_{\Bbbk}(\Bbbk[x_1, ..., x_n])$ for all $1 \leq i, j \leq n$. Thus one has a well-defined map

$$\rho : A_n(\Bbbk) \to \mathrm{End}_{\Bbbk}(\Bbbk[x_1, ..., x_n])$$

$$\sum_{k_1,...,k_n,j_1,...,j_n} c_{k_1\cdots k_n j_1\cdots j_n} x_1^{k_1} \cdots x_n^{k_n} y_1^{j_1} \cdots y_n^{j_n} \mapsto \sum_{k_1,...,k_n,j_1,...,j_n} c_{k_1\cdots k_n j_1\cdots j_n} (x_1)_l^{k_1} \cdots (x_n)_l^{k_n} (-\frac{\partial}{\partial x_1})^{j_1} \cdots (-\frac{\partial}{\partial x_n})^{j_n}$$

Clearly, $\rho$ is a $\Bbbk$-algebra morphism and $\mathrm{Im}\rho$ is the subalgebra of $\mathrm{End}_{\Bbbk}(\Bbbk[x_1, ..., x_n])$. In fact, $\mathrm{Im}\rho$ is just the subalgebra generated by $\Bbbk[x_1, ..., x_n]$ together with operators $\{\partial/\partial x_1, ..., \partial/\partial x_n\}$, which is called the **ring of differential operators with polynomial coefficients**. Write $\mathrm{Im}\rho$ as $\Delta(\Bbbk[x_1, ..., x_n])$. Then $\rho$ induces a surjective algebra morphism from $A_n(\Bbbk)$ to $\Delta(\Bbbk[x_1, ..., x_n])$, which is also injective since $A_n(\Bbbk)$ is simple. Thus, $\Delta(\Bbbk[x_1, ..., x_n]) \cong A_n(\Bbbk)$.

**Exercise 1.8.** Show that a simple ring may not be Artinian.

**Exercise 1.9.** Show that the Weyl algebra is not a completely reducible module over itself.

Let $\Bbbk$ be a field of positive characteristic $p$ and let $A_n(\Bbbk)$ be the $n$th Weyl algebra. By **Lemma 1.11**, for any $f \in Z(A_n(\Bbbk))$, one has $\partial f/\partial x_i = \partial f/\partial y_i = 0, \forall 1 \leq i \leq n$. Thus there exists a polynomial $g \in \Bbbk\langle x_1, ..., x_n, y_1, ..., y_n\rangle$ such that $f = g(x_1^p, ..., x_n^p, y_1^p, ..., y_n^p)$. It is clear that $x_1^p, ..., x_n^p, y_1^p, ..., y_n^p \in Z(A_n(\Bbbk))$, hence $Z(A_n(\Bbbk)) = \Bbbk[x_1^p, ..., x_n^p, y_1^p, ..., y_n^p]$. A similar argument can be applied to the characteristic zero case.

**Exercise 1.10.** If $\mathrm{char}\Bbbk = 0$, then the $n$th Weyl algebra over $\Bbbk$ is a central simple algebra, that is, $Z(A_n(\Bbbk)) = \Bbbk$. Thus $A_n(\Bbbk)$ is infinite dimensional over its center, which implies that $A_n(\Bbbk)$ is not P.I. by Kaplansky's Theorem.

**Exercise 1.11.** Let $A$ be an infinite dimensional simple $\Bbbk$-algebra(e.g. Weyl algebra $A_n(\Bbbk)$ with $\mathrm{char}\Bbbk = 0$). Show that any nonzero left module over $A$ is infinite dimensional.

**Exercise 1.12.** A $\mathbb{k}$-dervation of $A = \mathbb{k}[x_1, ..., x_n]$ is a $\mathbb{k}$-linear map $D : A \to A$ that satisfies Leibniz's law $D(ab) = aD(b) + D(a)b, \forall a, b \in A$. Denote $\mathrm{Der}_{\mathbb{k}} A$ as the set of all $\mathbb{k}$-dervations of $A$. Clearly, $\mathrm{Der}_{\mathbb{k}} A$ has a natural $A$-module structure. Show that $\mathrm{Der}_{\mathbb{k}} A$ is freely generated by $\{\partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_n\}$. That is,

$$\mathrm{Der}_{\mathbb{k}} A = A\frac{\partial}{\partial x_1} \oplus A\frac{\partial}{\partial x_2} \oplus \cdots \oplus A\frac{\partial}{\partial x_n}.$$

**Exercise 1.13.** Using the result of **Exercise 1.7**, show that $\mathrm{GKdim} A_n(\mathbb{k}) = 2n$, and hence

$$A_n(\mathbb{k}) \ncong A_m(\mathbb{k}), \forall n \neq m \in \mathbb{Z}_{\geq 1}.$$

# 2 Rings of Differential Operators on Algebraic Varieties

## 2.1 Derivation Rings

Throughout this section, $K$ denotes a commutative unital ring, $A$ denotes a commutative $K$-algebra and $\Bbbk$ denotes a field. The main object of study in this chapter is the derivation ring of a commutative $K$-algebra.

The set of all $K$-derivations of $A$ is denoted by $\mathrm{Der}_K A$. It is clear that $\mathrm{Der}_K A$ is an $A$-module via $(a\delta)(b) = a\delta(b)$ for all $\delta \in \mathrm{Der}_K A$ and $a, b \in A$. A straightforward verification shows that the commutator of two dervations is again a derivation. Thus this product gives $\mathrm{Der}_K A$ the structure of $K$-Lie algebra. As in **Exercise 1.12**, one has

**Proposition 2.1.** Let $A = K[x_1, ..., x_n]$ be the polynomial algebra. Then $\mathrm{Der}_K A$ is freely generated by

$$\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\}.$$

That is,

$$\mathrm{Der}_K A = A\frac{\partial}{\partial x_1} \oplus A\frac{\partial}{\partial x_2} \oplus \cdots \oplus A\frac{\partial}{\partial x_n}.$$

*Proof.* For any derivation $\delta \in \mathrm{Der}_K A$, a direct computation shows that

$$\delta = \sum_{k=1}^{n} \delta(x_k)\frac{\partial}{\partial x_k}.$$

Thus $\mathrm{Der}_K A$ can be generated by $\{\partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_n\}$ as an $A$-module. Suppose there are $f_1, ..., f_n \in A$ such that

$$\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} = 0.$$

Then $f_j = \sum\limits_{i=1}^{n} f_i(\partial/\partial x_i)(x_j) = 0, \forall 1 \leq j \leq n$, which follows that $\{\partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_n\}$ is $A$-linearly independent. Hence, we obtain that

$$\mathrm{Der}_K A = A\frac{\partial}{\partial x_1} \oplus A\frac{\partial}{\partial x_2} \oplus \cdots \oplus A\frac{\partial}{\partial x_n}.$$

□

Since one has natural embedding $A \to \mathrm{End}_K A, a \mapsto a_l$, one can identify $A$ with $\{a_l \in \mathrm{End}_K A | a \in A\}$.

**Definition 2.2** (Derivation ring)**.** The $K$-subalgebra of $\mathrm{End}_K A$ generated by $A$ and $\mathrm{Der}_K A$ is called the **derivation ring** of $A$, which is denoted by $\Delta(A)$.

**Remark 2.3.** It is convenient to consider not only $\Delta(A)$ but also the $K$-subalgebras of $\Delta(A)$ generated by $A$ and $\boldsymbol{d}$, where $\boldsymbol{d}$ is any $A$-submodule of $\mathrm{Der}_K(A)$ closed under Lie product(see **Exercise 2.1**). This subalgebra will be denoted by $A[\boldsymbol{d}]$, and hence $\Delta(A) = A[\boldsymbol{d}]$.

Since $\Delta(A)$ is a subalgebra of $\mathrm{End}_K A$, $A$ is naturally a left $\Delta(A)$-module. It is easy to see that $A$ is a cyclic left $\Delta(A)$-module generated by $1_A$. Set $\varphi : \Delta(A) \to A, u \mapsto u1_A$ and write $I = \mathrm{Ker}\varphi = \mathrm{ann}_{\Delta(A)}(1_A)$, then there is an exact sequence

$$0 \longrightarrow I \longrightarrow \Delta(A) \xrightarrow{\varphi} A \longrightarrow 0.$$

Clearly, the exact sequence splits, thus $\Delta(A) = I \oplus A$. It is also clear that $I \supseteq \mathrm{Der}_K A$, thus $I \supseteq \Delta(A)\mathrm{Der}_K A$. Suppose $I \subsetneq \Delta(A)\mathrm{Der}_K A$, by definition of $\Delta(A)$, there is $c \in A$ with $c \in I - \Delta(A)\mathrm{Der}_K A$. But $I \cap A = 0$, which forces $c$ to be zero. It follows that $0 \notin \Delta(A)\mathrm{Der}_K A$, which is a contradiction. Hence $I = \Delta(A)\mathrm{Der}_K A$ and we conclude that $\Delta(A) = \Delta(A)\mathrm{Der}_K A \oplus A$. It follows immediately that

**Proposition 2.4.** One has $A \cong \Delta(A)/\Delta(A)\mathrm{Der}_K A$ as left $\Delta(A)$-modules.

**Example 2.5.** If char$K = 0$, then $\Delta(K[x_1, ..., x_n]) \cong A_n(K)$.

*Proof.* It is obvious that there is a surjective $K$-algebra morphism from $A_n(K)$ to $\Delta(K[x_1, ..., x_n])$. Then the result holds since $A_n(K)$ is simple by **Theorem 1.12**. □

**Remark 2.6.** In general, for any affine variety $V \subseteq \Bbbk^n$, one can consider the derivation ring over $V$, that is,

$$\Delta\left(\frac{\Bbbk[x_1, ..., x_n]}{I(V)}\right).$$

Now let $\boldsymbol{d}$ be any $A$-submodule of $\mathrm{Der}_K A$ closed under the Lie bracket. Since $A[\boldsymbol{d}]$ is generated by $\boldsymbol{d}$ as an $A$-algebra, $A[\boldsymbol{d}]$ has a standard filtration over $A$ based on $\boldsymbol{d}$ as a generating set. More precisely, for each natural number $m$, put $A[\boldsymbol{d}]_m$ be the $A$-submodule spanned by all products of at most $m$ dervations from $\boldsymbol{d}$, then $\{A[\boldsymbol{d}]_m\}_{m\in\mathbb{N}}$ is the standard filtration of $A[\boldsymbol{d}]$. From now on, gr$A[\boldsymbol{d}]$ always refers to the associated graded ring of $A[\boldsymbol{d}]$ with respect to the standard filtration mentioned above. Then one has the following observation.

**Basic Observation.** The associated graded ring gr$A[\boldsymbol{d}]$ is commutative and there is a canonical surjection $S_A(\boldsymbol{d}) \to$ gr$A[\boldsymbol{d}]$, where $S_A(\boldsymbol{d})$ denotes the symmetric algebra of $\boldsymbol{d}_A$.

*Proof.* Take $\delta_1, \delta_2 \in \boldsymbol{d}, a \in A$, then one has $\delta_1 a - a\delta_1 = \delta_1(a)$ and $[\delta_1, \delta_2] \in \boldsymbol{d}$. The rest is clear now. □

**Remark 2.7.** Suppose an affine $K$-algebra $R$ has some standard finite dimensional filtration with respect to which gr$R$ is commutative, then $R$ is called an **almost commutative algebra**. In general, $A[\boldsymbol{d}]$ is not almost commutative[MR87, p. 571, 15.1.21].

**Corollary 2.8.** Suppose that $\boldsymbol{d}$ is finitely gnerated as an $A$-module. Then gr$A[\boldsymbol{d}]$ is a commutative affine $A$-algebra. Hence further if $A$ is Noetherian, then gr$A[\boldsymbol{d}]$ and $A[\boldsymbol{d}]$ are Noetherian.

*Proof.* By the result in **Exercise 2.2**, the first assertion is clear. The rest is clear by **Exercise 2.5**. □

**Exercise 2.1.** Consider the commutator of any two $K$-derivations of $A$: $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1, \delta_1, \delta_2 \in \mathrm{Der}_K A$. Show that $\mathrm{Der}_K A$ is closed under the bracket $[-,-]$ and this makes $\mathrm{Der}_K A$ into a $K$-Lie algebra.

**Exercise 2.2.** Suppose $M$ is a finitely generated $K$-module, then the tensor algebra of $M$ is an affine $K$-algebra.

**Exercise 2.3.** Let $S$ be a filtered ring. If gr$S$ is an integral domain, then $S$ is an integral domain.

**Exercise 2.4.** Let $S$ be a filtered ring. If gr$S$ is prime, then $S$ is prime, too.

**Exercise 2.5.** Let $S$ be a filtered ring. If gr$S$ is (left)right Noetherian, then so is $S$.

**Exercise 2.6.** Let $S$ be a filtered ring. If gr$S$ is (left)right Artinian, then so is $S$.

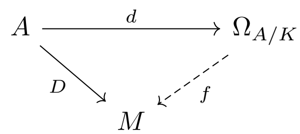
## 2.2 Kähler Differentials

Throughout this section, $K$ denotes a commutative unital ring, $A$ denotes a commutative $K$-algebra. For an $A$-module $M$, recall that a $K$-**dervation** from $A$ to $M$ is a $K$-linear map $\delta : A \to M$ satisfying Leibniz's law. The set of all $K$-derivations from $A$ to $M$ is denoted by $\mathrm{Der}_K(A, M)$. With this notation, $\mathrm{Der}_K(A, A) = \mathrm{Der}_K A$.

**Example 2.9.** Let $\alpha : A \to B$ be a $K$-algebra homomorphism between commutative $K$-algebras. Then for any $B$-module $M$, $\mathrm{Der}_K(A, M)$ has a natural $B$-module structure by defining $b\delta : A \to M, a \mapsto b\delta(a), \forall \delta \in \mathrm{Der}_K(A, M)$.

Now we shall introduce the module of Kähler differentials.

**Definition 2.10.** The **Kähler differentials**(or **Kähler differential module**) of $A$ is a pair $(\Omega(A), d)$ consisting of an $A$-module $\Omega(A)$ and a $K$-derivation $d : A \to \Omega(A)$ such that for any $A$-module $M$ and $K$-derivation $D : A \to M$, there exists a unique $A$-linear map $f : \Omega(A) \to M$ with $fd = D$. That is, the diagram




commutes. Here the derivation $d : A \to \Omega(A)$ is called the **universal derivation**. Sometimes, we also denote $\Omega(A)$ as $\Omega_K(A)$ to make the context clear.

**Remark.** The $r$-th exterior power $\wedge^r\Omega(A)$($\wedge$ is the wedge product **over** $A$) is denoted by $\Omega^r(A)$ and is called the **$r$-th Kähler differential forms**. The elements of $\Omega^r(A)$ are called **Kähler $r$-forms**.

Now we construct the Kähler differential module. Let $\Omega(A)$ be the $A$-module generated by the set $\{d(a)|a \in A\}$ subject to the relations

$$d(aa') = ad(a') + d(a)a', d(ka + k'a') = kd(a) + k'd(a'), \forall a, a' \in A, k, k' \in K.$$

And set $d : A \to \Omega(A), a \mapsto d(a)$. It is a quick check that $(\Omega(A), d)$ is the Kähler differential module of $A$. Sometimes, we also call $\Omega(A)$ the Kähler differential module of $A$. Since $(\Omega(A), d)$ exists, it must be unique up to isomorphism. By the definition of the Kähler differential module, the following fact is clear.

**Proposition 2.11.** Let $A$ be a commutative $K$-algebra and $(\Omega(A), d)$ be the Kähler differential module of $A$. Then we have the canonical isomorphism of $A$-modules $\mathrm{Der}_K(A, M) \cong \mathrm{Hom}_A(\Omega(A), M)$. In fact, this gives a natural isomorphism of $\mathrm{Der}_K(A, -)$ with $\mathrm{Hom}_A(\Omega(A), -)$. Thus, $\mathrm{Der}_K(A, -)$is a representable functor.

**Remark.** We emphasize the importance of the above proposition is that we can describe something nonlinear(derivations) in terms of something linear(module homomorphisms).

In particular, one has $\mathrm{Der}_K A \cong \Omega(A)^*$ as $A$-modules. We now specialize to the case of polynomial algebras. As in **Proposition 2.1**, one has the following.

**Proposition 2.12.** Let $A = K[x_1, ..., x_n]$ be the polynomial algebra. Then $\Omega_{A/K}$ is a finite free $K[x_1, ..., x_n]$-module with basis $\{dx_1, dx_2, ..., dx_n\}$. That is,

$$\Omega(A) = \bigoplus_{i=1}^{n} K[x_1, ..., x_n]dx_i.$$

*Proof.* By definition, $\Omega(A)$ is generated by $\{da|a \in A\}$, so $\Omega(A)$ can be generated by $\{dx_1, dx_2, ..., dx_n\}$ since $d$ is a $K$-derivation. It remains to show that $\{dx_1, dx_2, ..., dx_n\}$ is $A$-linearly independent. Suppose $\sum_{i=1}^{n} f_i dx_i = 0$ for some $f_i \in A$. Consider the partial derivation $\partial/\partial x_j : A \to A$, by the universal property of $(\Omega(A), d)$, there exists a unique $A$-linear map $\varphi_j : \Omega(A) \to A$ such that $\varphi_j d = \partial/\partial x_j$. Applying $\varphi_j$ to $\sum_{i=1}^{n} f_i dx_i = 0$ one finds $f_j = 0$. As $j$ is arbitrary we see that $\{dx_1, dx_2, ..., dx_n\}$ is $A$-linearly independent. □

**Remark.** It is not surprising that this result holds, since the free basis $\{\partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_n\}$ of $\mathrm{Der}_K(A, A)$ corresponds to the dual basis of $\{dx_1, ..., dx_n\}$(See **Proposition 2.1**). And it is a direct computation that for each $f \in A$, $df \in \Omega(A)$ can be expressed as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

**Corollary 2.13.** Let $A = K[x_1, ..., x_n]$ be the polynomial algebra. Then for each integer $1 \le r \le n$, the $r$-th Kähler differential forms $\Omega^r(A)$ is free $K[x_1, ..., x_n]$-module with basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_r} | 1 \le i_1 < \cdots < i_r \le n\}$. Thus

$$\Omega^r(A) = \bigoplus_{1 \le i_1 < \cdots < i_r \le n} K[x_1, ..., x_n]dx_{i_1} \wedge \cdots \wedge dx_{i_r}.$$

*Proof.* Recall that for any free $K$-module $V$ of rank $n$ with a basis $\{v_1, ..., v_n\}$, each exterior power $\wedge^r V (1 \leq r \leq n)$ has a basis $\{v_{i_1} \wedge \cdots \wedge v_{i_r} | 1 \leq i_1 < \cdots < i_r \leq n\}$. Thus the result holds. □

**Remark.** Given any n polynomials $f_1, ..., f_n \in A$, one has

$$df_1 \wedge df_2 \wedge \cdots \wedge df_n = J(f_1, f_2, ..., f_n)(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n) \in \Omega^n_{A/K},$$

where $J(f_1, f_2, ..., f_n) = \det(\partial f_i / \partial x_j)_{n \times n}$ is the Jacobian determinant of $f_1, f_2, ..., f_n$. This can be verified by a direct computation:

$$\begin{aligned} df_1 \wedge df_2 \wedge \cdots \wedge df_n &= \left( \sum_{i_1=1}^{n} \frac{\partial f_1}{\partial x_{i_1}} dx_{i_1} \right) \wedge \cdots \wedge \left( \sum_{i_n=1}^{n} \frac{\partial f_n}{\partial x_{i_n}} dx_{i_n} \right) \\ &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \frac{\partial f_1}{\partial x_{i_1}} \cdots \frac{\partial f_n}{\partial x_{i_n}} (dx_{i_1} \wedge \cdots dx_{i_n}) \\ &= \sum_{\sigma \in S_n} \frac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial f_n}{\partial x_{\sigma(n)}} (dx_{\sigma(1)} \wedge \cdots dx_{\sigma(n)}) \\ &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \frac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial f_n}{\partial x_{\sigma(n)}} (dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n) \\ &= J(f_1, f_2, ..., f_n)(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n). \end{aligned}$$

Sometimes, we write the Jacobian determinant $J(f_1, f_2, ..., f_n)$ as

$$\frac{df_1 \wedge df_2 \wedge \cdots \wedge df_n}{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}.$$

Next, we study a simple consequence of **Proposition 2.11**.

**Corollary 2.14.** Suppose that char$K = 0$ and that $\Omega(A)$ is free over $A$ on a basis $\{da_1, ..., da_n\}$, then:
(1)The set $\{a_1, ..., a_n\}$ is $K$-algebraic independent.
(2)The derivations $\partial/\partial a_i$ extend uniquely from $K[a_1, ..., a_n]$ to $A$.
(3)Der$_K A$ is free on the basis $\{\partial/\partial a_1, ..., \partial/\partial a_n\}$.
(4)Each derivation on $A$ is the unique extension of a derivation from $K[a_1, ..., a_n]$ to $A$.

*Proof.* (1)Suppose there exists a nonzero polynomial $f \in K[x_1, ..., x_n]$ with $f(a_1, ..., a_n) = 0$, without loss of generality, one may assume that $f$ is the polynomial which annihilates $(a_1, ..., a_n)$ with the least degree. By the definition of the Kähler differentials, there are derivations $\delta_1, ..., \delta_n$ having the property that $\delta_i(a_j) = \delta_{ij}$, the Kronecker delta. Applying any $\delta_i$ to $f$ one gets another polynomial identity of smaller degree. This observation shows that there is another nonzero polynomial over $K$ whose zero set contains $(a_1, ..., a_n)$, contradction.

(2)Keep notations in (1), one can easily see that $\delta_i$ is the extension of $\partial/\partial a_i$. Since $\Omega(A)$ is free, clearly the derivations $\partial/\partial a_i$ extend uniquely from $K[a_1, ..., a_n]$ to $A$.

(3)Abuse of notation, write $\delta_i$ as $\partial/\partial a_i$. Note that $D = \sum_{i=1}^{n} D(a_i)\delta_i, \forall D \in \mathrm{Der}_K A$, the result holds obviously.

(4)Clear. □

**Lemma 2.15** (The first fundamental exact sequence)**.** Let $A, B$ be commutative $K$-algebras, $\psi : A \to B$ a $K$-algebra homomorphisms and $M$ a $B$-module. Then: (1)$\mathrm{Hom}_B(B \otimes_A \Omega_K(A), M) \cong \mathrm{Der}_K(A, M)$ as $A$-modules. (2)There are exact sequences of $B$-modules

(i) $0 \longrightarrow \mathrm{Der}_A(B, M) \xrightarrow{\sigma} \mathrm{Der}_K(B, M) \xrightarrow{\tau} \mathrm{Der}_K(A, M)$

(ii) $B \otimes_A \Omega_K(A) \xrightarrow{\alpha} \Omega_K(B) \xrightarrow{\beta} \Omega_A(B) \longrightarrow 0$

with $\alpha$ being split injection if and only if $\tau$ is surjective for all ${}_B M$.

*Proof.* (1)One has the following natural isomorphism:

$$\mathrm{Hom}_B(B\otimes_A \Omega_K(A), M) \cong \mathrm{Hom}_A(\Omega_K(A), \mathrm{Hom}_B(B, M)) \cong \mathrm{Hom}_A(\Omega_K(A), M) \cong \mathrm{Der}_K(A, M).$$

(2)In **Example 2.9**, we have already seen that $\mathrm{Der}_K(A, M)$ has a natural $B$-module. Here we only prove (ii) since (i) is a direct check. Write $d_A : A \to \Omega_K(A), d_B : B \to \Omega_K(B), d'_B : B \to \Omega_A(B)$ for the universal derivations. By the universal property of $d_A$, one can define $\alpha : B \otimes_A \Omega_K(A) \to \Omega_K(B)$ via $\alpha(b \otimes d_A a) = bd_B(\psi(a))$, which is clearly a $B$-module homomorphism. For the same reason, one may define a $B$-module homomorphism $\beta : \Omega_K(B) \to \Omega_A(B)$ via $\beta(d_B b) = d'_B b$. Applying $\mathrm{Hom}_B(-, M)$ to the sequence of homomorphisms $B \otimes_A \Omega_K(A) \xrightarrow{\alpha} \Omega_K(B) \xrightarrow{\beta} \Omega_A(B) \longrightarrow 0$ , one obtains the following commutative diagram.

$$\begin{array}{ccccc}
\mathrm{Der}_A(B, M) & \xrightarrow{\sigma} & \mathrm{Der}_K(B, M) & \xrightarrow{\tau} & \mathrm{Der}_K(A, M) \\
\cong\uparrow & & \uparrow\cong & & \uparrow\cong \\
\mathrm{Hom}_B(\Omega_A(B), M) & \xrightarrow{\beta^*} & \mathrm{Hom}_B(\Omega_K(B), M) & \xrightarrow{\alpha^*} & \mathrm{Hom}_B(B \otimes_A \Omega_K(A), M)
\end{array}$$

The rest is clear now. □

**Corollary 2.16.** Let $A, B$ be commutative $K$-algebras, $\psi : A \to B$ a $K$-algebra homomorphisms. If $\Omega_K(A) = 0$, then $\Omega_K(B) \cong \Omega_A(B)$ as $B$-modules.

Next we shall show that when $A$ is $K$-affine and $K$ is Noetherian, the derivation ring $\Delta(A)$ must be Noetherian. Before that, we collect a few useful properties of $\mathrm{Der}_K A$.

**Lemma 2.17** (The second fundamental exact sequence)**.** Let $A$ be a commutative $K$-algebra, $I$ be a proper ideal of $A$. Then for any $A/I$-module $M$, there are exact sequences of $A$-modules:

(1) $0 \longrightarrow \mathrm{Der}_K(A/I, M) \xrightarrow{\alpha} \mathrm{Der}_K(A, M) \xrightarrow{\beta} \mathrm{Hom}_A(I, M)$;

(2) $0 \longrightarrow I/I^2 \xrightarrow{\beta'} A/I \otimes_A \Omega_K(A) \xrightarrow{\alpha'} \Omega_K(A/I)$.

*Proof.* (1) is a direct verification. To see (2), define $\beta'(a + I^2) = \bar{1} \otimes da$, $\alpha'(\bar{1} \otimes da) = d(\bar{a})$. Then one gets a sequence of $A/I$-module homomorphisms $0 \longrightarrow I/I^2 \xrightarrow{\beta'} A/I \otimes_A \Omega_K(A) \xrightarrow{\alpha'} \Omega_K(A/I)$. For any $A/I$-module $M$, applying $\mathrm{Hom}_{A/I}(-, M)$ to the above sequence, one gets the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Der}_K(A/I, M) & \xrightarrow{\alpha} & \mathrm{Der}_K(A, M) & \xrightarrow{\beta} & \mathrm{Hom}_A(I, M) \\
\cong\uparrow & & \uparrow\cong & & \uparrow\cong \\
\mathrm{Hom}_{A/I}(\Omega_K(A/I), M) & \xrightarrow{(\alpha')^*} & \mathrm{Hom}_{A/I}(A/I \otimes_A \Omega_K(A), M) & \xrightarrow{(\beta')^*} & \mathrm{Hom}_{A/I}(I/I^2, M)
\end{array}$$

The rest is clear now. □

**Proposition 2.18.** Let $A = K[x_1, ..., x_n]/I$ for some proper ideal $I \subseteq K[x_1, ..., x_n]$. Then there is an surjective $K[x_1, ..., x_n]$-module homomorphism $\{\delta \in \mathrm{Der}_K K[x_1, ..., x_n] | \delta(I) \subseteq I\} \to \mathrm{Der}_K A$. And if $K$ is Noetherian then $\mathrm{Der}_K A$ is a finitely generated $A$-module.

*Proof.* Write $\pi : K[x_1, ..., x_n] \to K[x_1, ..., x_n]/I$ for the natural projection. By **Lemma 2.17**, there is a canonical injection

$$\alpha : \mathrm{Der}_K(K[x_1, ..., x_n]/I) \to \mathrm{Der}_K(K[x_1, ..., x_n], K[x_1, ..., x_n]/I)$$
$$f \mapsto f\pi$$

One can also define an $K[x_1, ..., x_n]$-module homomorphism

$$\theta : \mathrm{Der}_K K[x_1, ..., x_n] \to \mathrm{Der}_K(K[x_1, ..., x_n], K[x_1, ..., x_n]/I), g \mapsto \pi g.$$

By using **Proposition 2.1**, one can easily show that $\theta$ is surjective. Then $\theta$ induces an $A$-module homomorphism $\tilde{\theta} : \{\delta \in \mathrm{Der}_K K[x_1, ..., x_n] | \delta(I) \subseteq I\} \to \mathrm{Der}_K A$ such that $\alpha\tilde{\theta} = \theta$. Since $\alpha$ is injective, it follows readily that $\tilde{\theta}$ is surjective. Now suppose $K$ is Noetherian, then $\{\delta \in \mathrm{Der}_K K[x_1, ..., x_n] | \delta(I) \subseteq I\}$ is a Noetherian $K[x_1, ..., x_n]$-module. So it follows that $\mathrm{Der}_K A$ is a finitely $A$-module. □

The proposition leads quickly to the following well-known result.

**Theorem 2.19.** If $K$ is Noetherian and $A$ is $K$-affine, then $\mathrm{gr}\Delta(A)$ is a commutative affine $K$-algebra and hence both $\mathrm{gr}\Delta(A)$ and $\Delta(A)$ are Noetherian.

*Proof.* By **Corollary 2.8**, it suffices to show that $\mathrm{Der}_K A$ is finitely generated as an $A$-module, but this is clear. □

**Exercise 2.7** (Another construction of the Kähler differentials)**.** Consider the multiplication map $\mu : A \otimes_K A \to A, a \otimes b \mapsto ab$, it is a $K$-algebra homomorphism since $A$ is commutative. Thus $I = \mathrm{Ker}\mu$ is an ideal of $A \otimes_K A$. We give $A \otimes_K A$ an $A$-module structure by $a \cdot (b \otimes c) = ab \otimes c$. Then clearly $I$ is an $A$-submodule of $A \otimes_K A$. Set $\Omega_K(A) = I/I^2$, it is naturally an $A$-module. Put $d : A \to I/I^2, a \mapsto (1 \otimes a - a \otimes 1) + I$,which is a $K$-linear map. Show that:
(1)The $K$-map $d : A \to \Omega_K(A)$ is a $K$-derivaion.
(2)The pair $(\Omega_{A/K}, d)$ constructed above is the Kähler differential module of $A$ and $\Omega_{A/K}$ is generated by $\{da | a \in A\}$ as an $A$-module.

## 2.3 Localization of the Kähler Differentials

In this section we study the localization properties of the Kähler differentials. Throughout this section, $K$ denotes a commutative unital ring, $A$ denotes a commutative $K$-algebra and $\Bbbk$ denotes a field.

**Lemma 2.20.** Let $S$ be a multiplicatively closed subset of $A$ and

$$t_S(A) = \{a \in R | \text{there exists } s \in S \text{such that } sa = 0\}$$

be the $S$-torsion submodule of $A$. Then for any $A$-module $M$ and $\delta \in \mathrm{Der}_K(A, M)$, we have
(1)$\delta(t_S(A)) \subseteq t_S(M)$.
(2)$\delta$ induces a canonical derivation in $\mathrm{Der}_K(A/t_S(A), M/t_S(M))$.
(3)$\delta$ induces a unique derivation $D$ in $\mathrm{Der}_K(A_S, M_S)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \longrightarrow & A_S \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle D} \\ M & \longrightarrow & M_S \end{array}$$

*Proof.* Since this lemma is a direct check, we only prove (3). Define $D : A_S \to M_S$ via

$$D(\frac{a}{s}) = \frac{\delta(a)s - a\delta(s)}{s^2}, \forall a \in A, s \in S.$$

Once we check that $D$ is well-defined, it follows readily that $D$ is the desired derivation. Suppose $a_1/s_1 = a_2/s_2$, then there is a $u \in S$ such that $u(s_2a_1 - s_1a_2) = 0$. Now we must show that there is a $v \in S$ such that it kills

$$s_2^2(\delta(a_1)s_1 - a_1\delta(s_1)) - s_1^2(\delta(a_2)s_2 - a_2\delta(s_2)).$$

In fact, one has

$$\begin{aligned} s_2^2(\delta(a_1)s_1 - a_1\delta(s_1)) - s_1^2(\delta(a_2)s_2 - a_2\delta(s_2)) &= s_1s_2^2\delta(a_1) - a_1s_2^2\delta(s_1) - s_1^2s_2\delta(a_2) + s_1^2a_2\delta(s_2) \\ &= -s_1s_2\delta(a_1s_2) - a_1s_2\delta(s_1s_2) - s_1s_2\delta(a_2s_1) + s_1a_2\delta(s_1s_2) \\ &= -s_1s_2\delta(a_1s_2 - a_2s_1) + (s_1a_2 - a_1s_2)\delta(s_1s_2). \end{aligned}$$

Now take $v = u^2$ and the rest is clear. □

The following corollary tells us that localization commutes with taking Kähler differentials.

**Corollary 2.21.** Let $S$ be a multiplicatively closed subset of $A$. Then there is an $A_S$-module isomorphism $\varphi : A_S \otimes_A \Omega_K(A) \to \Omega_K(A_S)$ which maps $1 \otimes d_A(a)$ into $d_{A_S}(as/s)$. In particular, $(\Omega_K(A))_S \cong \Omega_K(A_S)$.

*Proof.* Consider the map $\tau : \mathrm{Der}_K(B, M) \to \mathrm{Der}_K(A, M)$ is **Lemma 2.15**, and setting $B = A_S, \psi = \lambda_S : A \to A_S, a \mapsto as/s$, then for any $A_S$-module $M$, one has a canonical isomorphism

$$\begin{aligned} \xi_M : (M_S)_S &\to M \\ \frac{x/s}{t} &\mapsto \frac{x}{st} \end{aligned}$$

and hence one gets the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{1} & M \\ \downarrow & & \downarrow 1 \\ (M_S)_S & \xrightarrow{\xi_M} & M \end{array}$$

It follows readily that $\tau$ is surjective for any $A_S$-module $M$ by **Lemma 2.20**. By **Lemma 2.15**, we obtain the following exact sequence:

$$0 \longrightarrow A_S \otimes_A \Omega_K(A) \xrightarrow{\varphi} \Omega_K(A_S) \xrightarrow{\psi} \Omega_A(A_S) \longrightarrow 0$$

Note that for any $s, t \in S$ one has

$$0 = \delta(\frac{s}{s}) = \delta(\frac{t}{st} \cdot \frac{ts}{t}) = \frac{ts}{t}\delta(\frac{t}{st}), \forall \delta \in \mathrm{Der}_A(A_S, M),$$

thus it follows immediately that the universal derivation $d_{A_S} : A_S \to \Omega_A(A_S)$ is the zero map. Therefore $\Omega_A(A_S) = 0$ and the rest is clear now. ☐

Recall that for any modules $M, N$ over a commutative ring $R$, suppose $M$ is finitely presented, then for any multiplicatively closed subset of $R$, one has $(\mathrm{Hom}_R(M, N))_S \cong \mathrm{Hom}_{R_S}(M_S, N_S)$ as $R_S$-modules. So we obtain

**Corollary 2.22.** Let $K$ be Noetherian, $A$ be $K$-affine and $S$ be a multiplicatively closed subset of $A$. Then fot any $A$-module $M$, $A_S \otimes_A \mathrm{Der}_K(A, M) \cong \mathrm{Der}_K(A_S, M_S)$ as $A_S$-modules.

*Proof.* By condition, $A$ is Noetherian and $\Omega_K(A)$ is finitely generated as an $A$-module. In particular, $\Omega_K(A)$ is a finitely presented module. It follows that $(\mathrm{Hom}_A(\Omega_K(A), M))_S \cong \mathrm{Hom}_{A_S}((\Omega_K(A))_S, M_S) \cong \mathrm{Hom}_{A_S}(\Omega_K(A_S), M_S)$. The last isomorphism follows from **Corollary 2.21**. The result now follows by appyling **Proposition 2.11**. ☐

**Remark 2.23.** By a direct computation, one can deduce the fact that the isomorphism from $(\mathrm{Der}_K(A, M))_S$ to $\mathrm{Der}_K(A_S, M_S)$ maps $\delta/s$ to $\tilde{\delta}/s$, where

$$\begin{aligned} \tilde{\delta} : A_S &\to M_S \\ \frac{b}{t} &\mapsto \frac{\delta(b)t - \delta(t)b}{t^2}. \end{aligned}$$

Finally, we end this section by listing a theorem which gives connections with regular rings.

**Theorem 2.24.** Let $A$ be an affine domain over a field $\Bbbk$ of characteristic 0, then $A$ is regular if and only if $\Omega_{\Bbbk}(A)$ is projective. And if $A$ is regular, then both $\mathrm{Der}_{\Bbbk} A$ and $\Omega_{\Bbbk}(A)$ are finitely generated projective modules.

*Proof.* See [MR87, p.577, Corollary 2.11 and Theorem 2.12]. ☐

## 2.4 Rings of Differential Operators

Given a commutative algebra $R$ over a field of characteristic 0, we shall define its ring of differential operators $\mathcal{D}(R)$. This is a filtered $\Bbbk$-algebra in which

$$\mathcal{D}(R)_0 = \{f \in \mathrm{End}_{\Bbbk} R | f b_l - b_l f = 0, \forall b \in R\} = \{b_l \in \mathrm{End}_{\Bbbk} R | b \in R\} \cong R$$

and suppose we have already defined $\mathcal{D}(R)_{p-1}$ for $p \geq 1$, then we define

$$\mathcal{D}(R)_p = \{f \in \mathrm{End}_{\Bbbk} R | fb - bf \in \mathcal{D}(R)_{p-1} \text{ for all } b \in R\}.$$

Thus $\mathcal{D}(R) = \bigcup\limits_{p=0}^{\infty} \mathcal{D}(R)_p$. By definition, clearly one has $\mathcal{D}(R)_p \mathcal{D}(R)_0 \subseteq \mathcal{D}(R)_p, \forall p \geq 0$. By induction on $q \geq 0$, one can easily show that $\mathcal{D}(R)_p \mathcal{D}(R)_q \subseteq \mathcal{D}(R)_{p+q}, \forall p, q \geq 0$. Thus $\mathcal{D}(R)$ is indeed a filtered algebra.

**Lemma 2.25.** $\mathcal{D}(R)_1 = R + \mathrm{Der}_{\Bbbk} R$.

*Proof.* Clearly, one has $R + \mathrm{Der}_{\Bbbk} R \subseteq \mathcal{D}(R)_1$. Conversely, take $f \in \mathcal{D}(R)_1$, without loss of generality, assume that $f(1) = 0$, for otherwise one can relace $f$ by $f - f(1)$. Then for any $a, b \in R$, one has

$$f(ab) - af(b) = (af - fa)(b) = f(a)b,$$

which completes the proof. □

**Proposition 2.26.** The ring of differential operators $\mathcal{D}(R)$ has $\Delta(R)$ as a filtered subring.

# References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