Differential Core of Prime Ideals

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Let A be an algebra **over a field $\Bbbk$ of characteristic zero**. The aim of this note is to show that for any set of $\Bbbk$-derivations on A, say Δ, the Δ-core of any prime ideal of the $\Bbbk$-differential algebra (A, Δ) is still prime.

Recall that the **Δ-core** of an arbitrary ideal I in A is the largest Δ-ideal contained in I, which can be described as follows:

$$(I : \Delta) = \{a \in R | \delta_1 \cdots \delta_n(a) \in I, \forall \delta_1, ..., \delta_n \in I, n \geq 0\}.$$

Lemma ([Dix96])**.** Let (A, Δ) be a differential $\Bbbk$-algebra. Then $(P; \Delta)$ is prime for all prime ideals P of A.

Proof. To simplify notations, we shall write Q for $(P : \Delta)$. Let $a, b \in A$ such that $aAb \subseteq Q$, we must show that either a or b belongs to Q. Under this assumtion, we claim that

Claim. Let $\delta_1, ..., \delta_p \in \Delta, m_1, ..., m_p \in \mathbb{N}$ such that $\delta_1^{m_1} \cdots \delta_p^{m_p} b \notin P$, then $\delta_1^{n_1} \cdots \delta_p^{n_p} a \in P, \forall n_1, ..., n_p \in \mathbb{N}$.

Provide $\mathbb{N}^p$ with the following ordering $\leq$: (1) $(i_1, ..., i_p) < (j_1, ..., j_p)$ if $i_1 + \cdots + i_p < j_1 + \cdots + j_p$; (2) If $i_1 + \cdots + i_p = j_1 + \cdots + j_p$, then the ordering is defined by the lexicographic ordering. It is clear that $(\mathbb{N}^p, \leq)$ is a well-ordering set. The proof of the claim proceeds by transfinite induction in $(\mathbb{N}^p, \leq)$. Firstly, we can take the smallest element of $\mathbb{N}^p$ such that $\delta_1^{s_1} \cdots \delta_p^{s_p} b \notin P$. For any $x \in A$, write

$$\delta_1^{n_1+s_1} \cdots \delta_p^{n_p+s_p}(axb) = \sum_{\substack{i_k+j_k+l_k=n_k+s_k \\ 1\leq k\leq p}} \alpha(i_1, j_1, l_1, ..., i_p, j_p, l_p) \delta_1^{i_1} \delta_2^{i_2} \cdots \delta_p^{i_p}(a) \delta_1^{j_1} \delta_2^{j_2} \cdots \delta_p^{j_p}(x) \delta_1^{l_1} \delta_2^{l_2} \cdots \delta_p^{l_p}(b),$$

where $\alpha(i_1, j_1, l_1, ..., i_p, j_p, l_p) \in \mathbb{Z}_{\geq 1}$. Then one can rewrite the above expression as

$$\delta_1^{n_1+s_1} \cdots \delta_p^{n_p+s_p}(axb) = \alpha(n_1, s_1, ..., n_p, s_p) \delta_1^{n_1} \cdots \delta_p^{n_p}(a) x \delta_1^{s_1} \cdots \delta_p^{s_p}(b) + r,$$

where r is a sum of the form $\alpha(i_1, j_1, l_1, ..., i_p, j_p, l_p) \delta_1^{i_1} \delta_2^{i_2} \cdots \delta_p^{i_p}(a) \delta_1^{j_1} \delta_2^{j_2} \cdots \delta_p^{j_p}(x) \delta_1^{l_1} \delta_2^{l_2} \cdots \delta_p^{l_p}(b)$ such that $(i_1, ..., i_p) < (n_1, ..., n_p)$ or $(l_1, ..., l_p) < (s_1, ..., s_p)$. For $(n_1, ..., n_p) = (0, 0, ..., 0)$, it is easy to see $r \in P$, hence $ax\delta_1^{s_1} \cdots \delta_p^{s_p}(b) \in P$ since $axb \in Q$. It follows that $a \in P$ since x is arbitrary. By the induction hypothesis, one has $\delta_1^{i_1} \delta_2^{i_2} \cdots \delta_p^{i_p}(a) \in P, \forall (i_1, ..., i_p) < (n_1, ..., n_p)$. Thus $\alpha(n_1, 0, s_1, ...) \delta_1^{n_1} \cdots \delta_p^{n_p}(a) x \delta_1^{s_1} \cdots \delta_p^{s_p}(b) \in P$ by using the fact that $axb \in Q$. Since $\mathrm{char}\Bbbk = 0$ and x is arbitrary, it follows immediately that $\delta_1^{n_1} \cdots \delta_p^{n_p}(a) \in P$. This proves our claim.

Having established this, we finish by showing that $a \in Q$ if $b \notin Q$. For any $\delta_1, ..., \delta_p \in \Delta$, we must show that $\delta_1 \cdots \delta_p(a) \in P$. Since $b \notin Q$, there are $\delta_{p+1}, ..., \delta_t \in \Delta$ such that $\delta_1^0 \cdots \delta_p^0 \delta_{p+1}^1 \cdots \delta_t^1(b) \notin P$. From our claim, we have $\delta_1^1 \cdots \delta_p^1(a) = \delta_1^1 \cdots \delta_p^1 \delta_{p+1}^0 \cdots \delta_t^0(a) \in P$. □

References

[Dix96] Jacques Dixmier. *Enveloping algebras*. Number 11. American Mathematical Soc., 1996.

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